1. Option Pricing and the Black Scholes equation

a) $(\Omega, \mathcal{F}, \mathbb{P})$ $\mathcal{G} \subseteq \mathcal{F}$ Sub σ-algebra s.t. X is $\mathcal{G}$-measurable and Z is independent of $\mathcal{G}$. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be Borel-measurable and bounded. Define

$$h(x) = \mathbb{E}[f(x,Z)] \text{ for all } x \in \mathbb{R}$$

(i) We show that $\mathbb{E}[f(X,Z)|\mathcal{G}] = h(X)$ almost surely

where $f(x,y) = \mathbb{1}_{A \times B}(x,y) = \mathbb{1}_A(x)\mathbb{1}_B(y)$ $\quad A, B \in \mathcal{B}(\mathbb{R})$

$$\mathbb{E}[f(X,Z)|\mathcal{G}] = \int_{\mathbb{R}^2} \mathbb{1}_A(x)\,\mathbb{1}_B(z)\,\frac{d\mathbb{P}|\mathcal{G}}{dx\,dz}\,d\sigma$$

independence

$$= \int_A \int_B \frac{d}{dx\,dz}\left(\frac{\mathbb{P}\cdot\mathcal{G}}{\mathcal{G}}\right) dx\,dz = \int_A \int_B \frac{d\mathbb{P}}{dx\,dz}\,dx\,dz = \mathbb{E}[f(x,z)] \quad \square$$

(ii) Now let $f: \mathbb{R}^2 \to \mathbb{R}$ bounded, Borel-measurable

there exists $f^n(x,y) = \sum_{i=1}^{k_n} \underset{\in \mathbb{R}}{a_i^{(n)}}\, \mathbb{1}_{\underset{\in \mathcal{B}(\mathbb{R})}{A_i^{(n)} \times B_i^{(n)}}}\underset{\in \mathbb{R}^2}{(x,y)}$ s.t. $f^n \to f$ with $|f^n| \leq |f|$ $\forall (x,y) \in \mathbb{R}^2$

then $\mathbb{E}[f(X,Z)|\mathcal{G}] = \mathbb{E}[\lim_{n\to\infty} f^n(X,Z)|\mathcal{G}] = \mathbb{E}[\lim_{n\to\infty} \sum_{i=1}^{k_n} a_i^{(n)}\, \mathbb{1}_{A_i^{(n)} \times B_i^{(n)}}(X,Z)|\mathcal{G}]$

$$\overset{\text{lin. of } \mathbb{E}}{=} \lim_{n\to\infty} \sum_{i=1}^{k_n} \mathbb{E}[a_i^{(n)}\, \mathbb{1}_{A_i^{(n)} \times B_i^{(n)}}(X,Z)|\mathcal{G}]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{k_n} a_i \cdot \mathbb{E}[\mathbb{1}_{A_i^{(n)} \times B_i^{(n)}}(X,Z)] = \mathbb{E}[\lim_{n\to\infty} \sum_{i=1}^{k_n} a_i \cdot \mathbb{1}_{A_i^{(n)} \times B_i^{(n)}}(X,Z)] = \mathbb{E}[f(X,Z)]$$

$$= h(X)$$

b) Let W one-dimensional BM $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ filtration

Black-Scholes stock price process: $S_t = S_0 \exp\left((r - \frac{1}{2}\sigma^2)t + \sigma W_t \right) \quad t \in [0,T]$

Let $g : \mathbb{R}_+ \to \mathbb{R}$ be Borel-measurable s.t. $\mathbb{E}[|g(S_T)|] < \infty$

The no-arbitrage price at time $t \in [0,T]$ of a European claim with payoff $g(S_T)$ at maturity T is

$$V_t = e^{-r(T-t)} \mathbb{E}[g(S_T)|\mathcal{F}_t] \quad t \in [0,T]$$

(i) European <u>digital</u> call option $g(S_T) = \mathbb{1}_{\{S_T \geq K\}}$ $\quad K > 0$ strike

for it we have
$$V_t = e^{-r(T-t)} \mathbb{E}[\mathbb{1}_{\{S_T \geq K\}} | \mathcal{F}_t]$$

$$\mathbb{P}_{S_t}(S_T \geq K | \mathcal{F}_t) = \mathbb{P}\left(S_t \cdot \exp\left((r - \tfrac{1}{2}\sigma^2)(T-t)\right) + (W_T - W_t)\sigma\right) \geq K | \mathcal{F}_t)$$

$$= \mathbb{P}_{S_t}\left(W_T - W_t \geq \frac{\ln\left(\frac{K}{S_t}\right) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma} \Big| \mathcal{F}_t \right)$$

$$= \mathbb{P}_{S_t}\Big(\underbrace{-Z}_{\sim N(0,1)} \geq \frac{-\ln\left(\frac{S_t}{K}\right) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \Big| \mathcal{F}_t \Big)$$

indep. of $\mathcal{F}_t$ $\Rightarrow$ can be dropped (see a)

$$\overset{(a)}{=} \mathbb{P}_s\Big(\frac{\ln\left(\frac{s}{K}\right) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \geq \underbrace{Z}_{\sim N(0,1)} \Big)$$

$$= \Phi\left(\frac{\ln\left(\frac{s}{K}\right) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right)$$

Due to BM definition (i.i.d $W_T - W_t \sim N(0, T-t)$) we have

$$\mathbb{E}[\mathbb{1}_{\{S_T \leq K\}} | \mathcal{F}_t] = \mathbb{P}[S_T \geq K | \mathcal{F}_t] = \Phi\left(\frac{\ln\left(\frac{s}{K}\right) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right)$$

$\square$

(ii) Showing that v satisfies the BS-PDE

remember by heart

Compute
$$\partial_t v(t,s) = r e^{-r(T-t)} \Phi\left(\frac{\ln(s/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right)$$
$$+ e^{-r(T-t)} \cdot \frac{1}{\sqrt{2\pi}} \left(\exp\left(-\frac{1}{2} \frac{(\ln(s/K) + (r - \frac{1}{2}\sigma^2)(T-t))^2}{\sigma^2 (T-t)} \right) - 1 \right) \left(-\frac{r - \frac{1}{2}\sigma^2}{\sigma\sqrt{T-t}} (T-t) \right)$$

$$\partial_S V(t,S) = e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} \left(\exp\left(-\frac{1}{2} \frac{(\ln(S/K) + (r - \frac{1}{2}\sigma^2)(T-t))^2}{\sigma^2 (T-t)} \right) - 1 \right) \left(\frac{\frac{1}{S}}{\sigma\sqrt{T-t}} \right)$$

$$\partial_{SS} V(t,S) = e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} \left(\exp\left(-\frac{1}{2} \frac{(\ln(S/K) + (r - \frac{1}{2}\sigma^2)(T-t))^2}{\sigma^2 (T-t)} \right) \right) \left(\frac{\frac{1}{S^2}}{\sigma^2 (T-t)} - \frac{\frac{1}{S^2}}{\sigma\sqrt{T-t}} \right) - e^{-r(T-t)} \cdot \frac{1}{\sqrt{2\pi}} \frac{\frac{1}{S^2}}{\sigma\sqrt{T-t}}$$

Plugging the found terms into the Black-Scholes PDE:

$$r e^{-r(T-t)} \Phi\left(\frac{\ln(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right)$$

$$+ e^{-r(T-t)} \cdot \frac{1}{\sqrt{2\pi}} \left(\exp\left(-\frac{1}{2} \frac{(\ln(S/K) + (r - \frac{1}{2}\sigma^2)(T-t))^2}{\sigma^2 (T-t)} \right) - 1 \right) \left(- \frac{r - \frac{1}{2}\sigma^2}{\sigma\sqrt{T-t}} (T-t) \right)$$

$$+ \frac{1}{2}\sigma^2 S^2 e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} \left(\exp\left(-\frac{1}{2} \frac{(\ln(S/K) + (r - \frac{1}{2}\sigma^2)(T-t))^2}{\sigma^2 (T-t)} \right) \right) \left(\frac{\frac{1}{S^2}}{\sigma^2 (T-t)} - \frac{\frac{K}{S^2}}{\sigma\sqrt{T-t}} \right) - e^{-r(T-t)} \cdot \frac{1}{\sqrt{2\pi}} \frac{\frac{1}{S^2}}{\sigma\sqrt{T-t}}$$

$$+ r \cdot S \cdot e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} \left(\exp\left(-\frac{1}{2} \frac{(\ln(S/K) + (r - \frac{1}{2}\sigma^2)(T-t))^2}{\sigma^2 (T-t)} \right) - 1 \right) \left(\frac{\frac{1}{S}}{\sigma\sqrt{T-t}} \right)$$

$$- r \cdot e^{-r(T-t)} \Phi\left(\frac{\ln(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right)$$

$$v(t,s) = \underbrace{e^{-r(T-t)}}_{D}\,\Phi\left(\frac{\overbrace{\ln(s/K)}^{A} + \overbrace{(r-\frac{1}{2}\sigma^2)(T-t)}^{B}}{\underbrace{\sigma\sqrt{T-t}}_{C}}\right),$$

$$E(y) := \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

$$\partial_t v(t,s) = rD\,\Phi\left(\frac{A+B}{C}\right) + D\cdot E\left(\frac{A+B}{C}\right)\cdot\left(-\frac{B}{\sigma}\cdot\frac{1}{2\sqrt{T-t}}\right)$$

$$\partial_s v(t,s) = D\cdot E\left(\frac{A+B}{C}\right)\cdot\left(\frac{1}{s\cdot C}\right)$$

$$\partial^2_{ss} v(t,s) = D\cdot E\left(\frac{A+B}{C}\right)\cdot\left(-\frac{A+B}{C}\right)\left(\frac{1}{sC}\right)^2 + DE\left(\frac{A+B}{C}\right)\left(-\frac{1}{s^2C}\right)$$

Plug them in all together in the Black-Scholes ODE:

$$rD\,\Phi\left(\frac{A+B}{C}\right) + DE\left(\frac{A+B}{C}\right)\cdot\left(-\frac{B}{\sigma}\frac{1}{2\sqrt{T-t}}\right) + \frac{1}{2}\sigma^2 s^2\left(DE\left(\frac{A+B}{C}\right)\left(-\frac{A+B}{C}\right)\left(\frac{1}{sC}\right)^2 + DE\left(\frac{A+B}{C}\right)\left(-\frac{1}{s^2C}\right)\right)$$

$$+ r\cdot s\cdot DE\left(\frac{A+B}{C}\right)\left(\frac{1}{sC}\right) - r\cdot D\,\Phi\left(\frac{A+B}{C}\right)$$

$$= DE\left(\frac{A+B}{C}\right)\left(-\frac{B}{\sigma}\frac{1}{2\sqrt{T-t}}\right) - DE\left(\frac{A+B}{C}\right)\frac{A+B}{C^3}\frac{\sigma^2}{2} - DE\left(\frac{A+B}{C}\right)\frac{\sigma^2}{2C}$$

$$+ rDE\left(\frac{A+B}{C}\right)\frac{1}{C}$$

$$= DE\left(\frac{A+B}{C}\right)\left(-\frac{B}{\sigma}\frac{1}{2\sqrt{T-t}}\right) - DE\left(\frac{A+B}{C}\right)\left(\frac{A}{\sigma(T-t)\sqrt{T-t}\cdot 2} + \frac{B}{2\sigma\sqrt{T-t}}\right) - DE\frac{\sigma}{2\sqrt{T-t}}$$

$$+ DE\left(\frac{A+B}{C}\right)\frac{r}{C}$$

$$= -DE\left(\frac{A+B}{C}\right)\left(\frac{A}{\sigma(T-t)\sqrt{T-t}\cdot 2}\right) \quad ? \quad \neq 0$$

(iii)

$$\lim_{t\uparrow T} \frac{\ln\left(\frac{s}{K}\right) + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \begin{cases} -\infty & \text{if } s<K \\ 0 & \text{if } s=K \quad (\text{L'Hôpital} \\ \infty & \text{if } s>K \end{cases}$$

Thus, the limit doesn't exist.

c) Show that the Black Scholes equation can be reduced to the classic heat equation: $\partial_\tau u - \partial^2_{xx} u = 0$

$\tau = T - t \qquad x = \ln(S/K)$

$W(\tau, x) = V(T-\tau, \exp(x)\cdot K)$

$\partial_t V = -\partial_t W(\tau, x) \quad \Big| \quad \partial_S V = \partial_x W(\tau, x)\cdot \frac{1}{S} \quad \Big| \quad \partial_{SS}^2 V = \partial_{xx}^2 W(\tau,x)\cdot\frac{1}{S^2} - \partial_x W(\tau,x)\frac{1}{S^2}$

$= \frac{1}{S^2}\left(\partial_{xx}^2 W - \partial_x W\right)$

Plug into (3):

$-\partial_\tau W + \frac{1}{2}\sigma^2 S^2 \frac{1}{S^2}\cdot\left(\partial_{xx}^2 W - \partial_x W\right) + r\cdot S\cdot\partial_x W\cdot\frac{1}{S} - r\cdot W$

$= -\partial_\tau W + \frac{1}{2}\sigma^2\partial_{xx}^2 W + \partial_x W\left(r - \frac{1}{2}\sigma^2\right) - r\cdot W$

Define $u(\tau, x) = e^{a\tau + bx} W(\tau, x) \Leftrightarrow W(\tau, x) = \underbrace{e^{-a\tau - bx}}_{:=A} u(\tau, x)$

$\partial_\tau W = A\cdot(-a u + \partial_\tau u)$

$\partial_x W = A\cdot(-b u + \partial_x u)$

$\Big|\ \partial_{xx}^2 W = A\left(b^2 u - b\partial_x u - b\partial_x u + \partial_{xx}^2 u\right)$

$= A\left(b^2 u - 2b\partial_x u + \partial_{xx}^2 u\right)$

Plug in:

$+au - \partial_\tau u + \frac{1}{2}\sigma^2\left(b^2 u - 2b\partial_x u + \partial_{xx}^2 u\right) + (-bu + \partial_x u)(r - \frac{1}{2}\sigma^2) - ru$

$u: \ a + \frac{1}{2}\sigma^2 b^2 - b\ (r - \frac{1}{2}\sigma^2) - r = 0$

$\partial_x u: \ -\sigma^2 b + (r - \frac{1}{2}\sigma^2) = 0 \Leftrightarrow b = \frac{r}{\sigma^2} - \frac{1}{2}$

$a + \frac{1}{2}\sigma^2\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 - \left(\frac{r}{\sigma^2} - \frac{1}{2}\right)\left(r - \frac{1}{2}\sigma^2\right) - r = 0$

$\Leftrightarrow a = \frac{(2r + \sigma^2)^2}{8\sigma^2}$

Then we get: $-\partial_\tau u + \frac{1}{2}\sigma^2\partial_{xx}^2 u = 0 \Leftrightarrow \partial_\tau u - \frac{1}{2}\sigma^2\partial_{xx}^2 u = 0$

define $\tau = \frac{1}{2}\sigma^2\tau'$, because then $u(\tau, x) = u(\frac{1}{2}\sigma^2\tau', x)$

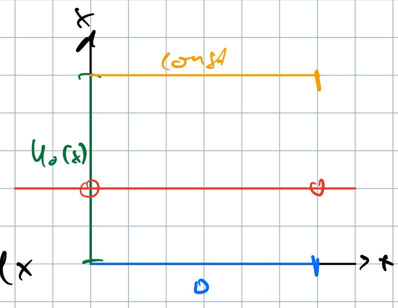
$\frac{\partial u}{\partial \tau'} = \frac{\partial u}{\partial \tau}\cdot\frac{\partial \tau}{\partial \tau'} = \partial_\tau u\cdot\frac{1}{2}\sigma^2$

$\Rightarrow \partial_{\tau'} u - \partial_{xx}^2 u = 0$ which is exactly the heat equation $\square$

2. Finite-Difference method for the heat equation.

a) f is 1 times continuously differentiable because $u \in C^{1,2}([0,1]^2)$ on $[0,1]$

We show that $\frac{df}{dt} \leq 0 \quad \forall t \in [0,1]$



$$\frac{df}{dt} = \int_0^1 2\, u(x,t) \cdot \frac{\partial u(x,t)}{\partial t}\, dx = 2\int_0^1 u(x,t) \frac{\partial^2 u(x,t)}{\partial^2 x}\, dx$$

$$\int_0^1 u(x,t) \frac{\partial^2 u(x,t)}{\partial^2 x}\, dt = \left[u(x,t) \frac{\partial u(x,t)}{\partial x} \right]_0^1 - \int_0^1 \left(\frac{\partial u(x,t)}{\partial x} \right)^2 dx$$

$$= u(1,t)\frac{\partial u(1,t)}{\partial x} - u(0,t)\frac{\partial u(0,t)}{\partial x} - \int_0^1 \left(\frac{\partial u(x,t)}{\partial x} \right)^2 dx = 0 - 0 - \int_0^1 \left(\frac{\partial u(t,t)}{\partial x} \right)^2 dx \leq 0 \quad \square$$

We show the uniqueness of u if it exists through a contradiction.

Suppose $u, v \in C_1^2([0,1]\times[0,1])$ solve (4) where $u \neq v$. Let $u_0 \in C^2((0,1))$

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = 0 \iff \frac{\partial u}{\partial t} - \frac{\partial v}{\partial t} = 0 \text{ and } \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 v}{\partial x^2} = 0$$

$$\iff \frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} \text{ and } \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2}$$

on $(0,1)\times(0,1)$

$$\Rightarrow u = v + K_x \quad \text{integrate and } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} + A_t$$

$$\iff u = v + K_x \text{ and } u = v + A_t x + B_t$$

$$\Rightarrow \underline{u = v + A_t x}$$

Applying BC:

$$u(x,0) = v(x,0) + A_t x = u_0 \iff A_t = \frac{u_0 - v(x,0)}{x} \quad \forall x \in (0,1)$$

So: $u(x,t) = v(x,t) + u_0(x) - v(x,0)$

and $v(x,0) = u_0(x) \Rightarrow u(x,t) = v(x,t)$ which is in contradiction to the initial assumption that $u(x,t) \neq v(x,t)$. We conclude that there exist no two or more unique solutions, thus at most one.

b) $u \in C_2^4([0,1]\times[0,1])$

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0 & \text{on } (0,1)\times(0,1), \\ u(x,0) = u_0(x) & \forall x \in (0,1), \\ u(0,t) = \dfrac{\partial u}{\partial x}(1,t) = 0 & \forall t \in (0,1). \end{cases}$$

We apply Taylor to u:

$$u(x+h, t+k) = u(x,t) + \partial_x u(x,t) h + \partial_t u(x,t) k + \frac{1}{2}\partial_{xx}^2 u(x,t) h^2 + \partial_{xy}^2 u(x,t) h k + \frac{1}{2} \partial_{yy}^2 u(x,t) k^2 + O(h^3 + k^2)$$

where $x = ih$, $y = mk$

Or in 1D: $u(x+h,t) = u(x,t) + \partial_x u(x,t) h + \frac{1}{2}\partial_{xx}^2 u(x,t) h^2 + O(h^2)$

$u(x, t+k) = u(x,t) + \partial_t u(x,t) k + \frac{1}{2}\partial_{tt}^2 u(x,t) k^2 + O(k^2)$

$$\frac{\partial u}{\partial t}\Big|_{(x,t)=(x_i,t_m)} \overset{\text{Taylor solve for } \partial_t u}{=} \frac{u(x_i,t_m+k)-u(x_i,t_m)}{k} + \frac{1}{2}\partial_{tt}^2 u[x_i,t_m]\frac{k^2}{k} + \frac{O(k^2)}{k}$$

$$= \frac{u(x_i,t_m+k)-u(x_i,t_m)}{k} + O(k)$$

Taylor to right and left:

$$u(x_N+h,t_m) = u(x_N,t_m) + \partial_x u(x_N,t_m)h + \frac{1}{2}\partial_{xx}^2 u(x_N,t_m)h^2 + \frac{1}{6}\partial_{xxx}^3 u(x_N,t_m)h^3 + O(h^3)$$

$$u(x_N-h,t_m) = u(x_N,t_m) - \partial_x u(x_N,t_m)h + \frac{1}{2}\partial_{xx}^2 u(x_N,t_m)h^2 - \frac{1}{6}\partial_{xxx}^3 u(x_N,t_m)h^3 + O(h^3)$$

Solve the system of equations for $\partial_x u(x_N,t_m)$

$$\partial_x u(x_N,t_m) = \frac{u(x_N+h,t_m) - u(x_N-h,t_m) - \frac{1}{3}\partial_{xxx}^3 u(x_N,t_m) + O(h^3)}{2h}$$

$$= \frac{u(x_N+h,t_m) - u(x_N-h,t_m)}{2h} + O(h^2)$$

Previous system solve for $\partial_{xx}^2 u(x_N,t_m)$

$$\partial_{xx}^2 u(x_N,t_m) = \frac{u(x_N+h,t_m) - 2u(x_N,t_m) + u(x_N-h,t_m) + O(h^4)}{h^2}$$

$$= \frac{u(x_N+h,t_m) - 2u(x_N,t_m) + u(x_N-h,t_m)}{h^2} + O(h^2)$$

c)

$$\begin{cases} \dfrac{u_i^{m+1}-u_i^m}{k} - \dfrac{u_{i-1}^m - 2u_i^m + u_{i+1}^m}{h^2} = 0, & \text{for } 0 < i \leq N, \\ \dfrac{-u_{N-1}^{m+1} + u_{N+1}^{m+1}}{2h} = 0, & \\ u_0^{m+1} = 0, & \end{cases} \qquad (5)$$

Goal: Eliminate u_0^m, u_{N+1}^m

$$\frac{u_1^{m+1}-u_1^m}{k} - \frac{\cancelto{0}{u_0^m} - 2u_1^m + u_2^m}{h^2} = 0 \qquad \text{for } i=1$$

$$\text{I} \quad \frac{u_N^{m+1}-u_N^m}{k} - \frac{u_{N-1}^m - 2u_N^m + u_{N+1}^m}{h^2} = 0$$

$$\text{II} \quad -u_{N-1}^m + u_{N+1}^m = 0$$

$$\text{I}+\text{II}: \quad \frac{u_N^{m+1}-u_N^m}{k} - \frac{2u_{N-1}^m -}{h^2} = 0$$

$$u_i^{m+1} = u_i^m + k\frac{u_{i-1}^m - 2u_i^m + u_{i+1}^m}{h^2}$$

$$= \frac{k}{h^2}u_{i-1}^m + \left(1-\frac{2k}{h^2}\right)u_i^m + u_{i+1}^m$$

$$\begin{bmatrix} u_1^{m+1} \\ u_2^{m+1} \\ \vdots \\ u_N^{m+1} \end{bmatrix} = \begin{bmatrix} \left(1-2\frac{k}{h^2}\right) & \frac{k}{h^2} & 0 & \cdots & & 0 \\ \frac{k}{h^2} & \left(1-\frac{2k}{h^2}\right) & 1 & 0 & \cdots & 0 \\ 0 & \frac{k}{h^2} & \left(1-\frac{2k}{h^2}\right) & 1 & 0 \cdots & 0 \\ \cdots & & & & & \\ & & & & 2\frac{k}{h^2} & \left(1-\frac{2k}{h^2}\right) \end{bmatrix} \begin{bmatrix} u_1^m \\ u_2^m \\ \vdots \\ u_N^m \end{bmatrix}$$

$$= \underline{\underline{I}} - \frac{k}{h^2}\underline{\underline{G}} \qquad \square$$

d) Similarly